MATH 1A - FINAL EXAM DELUXE

PEYAM RYAN TABRIZIAN

Name: ___________________

Instructions: This is it, people! Your final hurdle to freedom :) This exam counts for 50% of your grade and you officially have 110 minutes to take this exam (although I will try to give you more time). Please $\boxed{\text{box}}$ your answers!

By the way, enjoy the rest of your $\sum$mer :)

Note: This is the final exam deluxe, **NOT** the regular final exam. Please sign here to acknowledge this fact: ________________________.

1		10
2		10
3		10
4		15
5		20
6		15
7		20
8		30
9		10
10		10
Bonus 1		5
Bonus 2		5
Bonus 3		5
Total		150

Date: Friday, August 12th, 2011.

1. *(10 points, 5 points each)* Find the following limits

(a) $\lim_{x\to\infty} \sqrt{x^2+1} - x$

(b) $\lim_{x\to 0^+} x^{x^2}$

2. *(10 points)* Use the **definition** of the derivative to calculate $f'(x)$, where:

$$f(x) = x^2$$

3. *(10 points, 5 points each)* Find the derivatives of the following functions
   (a) y', where $x^y = y^x$

   Hint: Take $\ln$s first and then differentiate!

(b) y' at $(0,0)$, where $\sin(y) = x^2 - y^2$

4. *(15 points)* Assume Peyam's happiness function is given by:

$$H = M^2L + 2G$$

Where:

- M is the happiness due to holding office hours
- L is the happiness due to lecturing
- G is the happiness due to grading exams

Assume that at the end of the summer:

- Peyam's happiness due to holding office hours is 5 utils/week, and is decreasing by 2 utils/week
- Peyam's happiness due to lecturing is 10 utils/week, and is decreasing by 1 util/week
- Peyam's happiness due to grading exams, is 2 utils/week, and is decreasing by 1 utils/week.

Question: By how much is Peyam's happiness increasing/decreasing at the end of the summer?

(This page is left blank for you to have more space to work on question 4.)

5. *(20 points)* What is the area of the largest rectangle that can be put inside the parabola $y = 4 - x^2$?

   Note: In the final step, the algebra is a bit messy!

(This page is left blank for you to have more space to work on question 4.)

6. *(15 points)*

(a) *(13 points)* Show that the following equation has exactly one solution:

$$\cos(x) = 2x$$

(b) *(2 points)* Use part (a) to show that the following function has exactly one critical point:

$$g(x) = \sin(x) - x^2$$

7. *(20 points)* Use the **definition** of the integral to evaluate:

$$\int_0^1 x^3 dx$$

You may use the following formulas:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Note: -2 for not writing $\lim_{n\to\infty}$

8. *(30 points, 5 points each)* Find the following:

(a) $\int_{-1}^{1} \sqrt{1-x^2}dx$

Note: Don't spend too much time on this one, either you know it or you don't!

(b) The antiderivative F of $f(x) = 3e^x + 4\sec^2(x)$ which satisfies $F(0) = 1$.

(c) $g'(x)$, where $g(x) = \int_{x^2}^{e^x} \sin(t^3)dt$

(d) $\int (\cos(x))^3 \sin(x)dx$

(e) $\int_{e}^{e^2} \left(\frac{(\ln(x))^3}{x} \right) dx$

(f) The average value of $f(x) = \sin(x^5) \left(\cos(x^2) + e^{x^2} + x^4 \right)$ on $[-\pi, \pi]$

9. *(10 points)* Find the area of the region enclosed by the curves:

$y = \cos(x)$ and $y = -\cos(x)$ from 0 to π

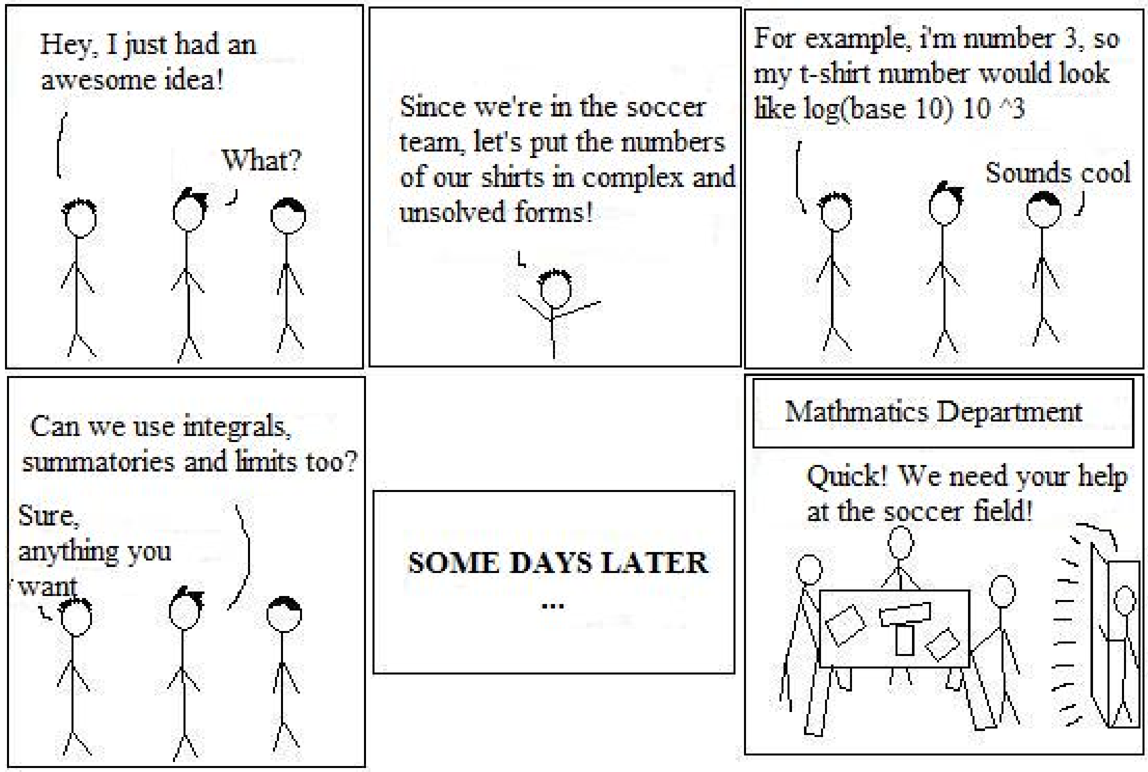
Hint: It might help to notice a certain symmetry in your picture!

10. *(10 points)* If $f(x) = \frac{x^3}{3} - \frac{x^2}{2}$, find:

(a) Intervals of increase and decrease, and local max/min

(b) Intervals of concavity and inflection points (just give me the $x-$ coordinate of the I.P.)

1A/Practice Exams/Soccer.jpg



Bonus 1 *(5 points)* Fill in the gaps in the following proof that the function f is not integrable on $[0, 1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Step 1: Pick x_i^* such that ________________________. Then:

$\int_0^1 f(x)dx =$

Step 2: Pick x_i^* such that ________________________. Then:

$\int_0^1 f(x)dx =$

Since we get two different answers for the integral, we have a contradiction. $\Rightarrow\Leftarrow$. And hence f is not integrable on $[0, 1]$.

Note: See the handout 'Integration sucks!!!' for a nice discussion of this problem!

Bonus 2 *(5 points)* Another way to define $\ln(x)$ is:

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Show **using this definition only** that $\ln(e^x) = x$.

Hint: Let $g(x) = \ln(e^x) = \int_1^{e^x} \frac{1}{t} dt$. Differentiate g, simplify, and antidifferentiate. Make sure you face the issue of the constant!

Bonus 3 *(5 points)* Define the **Product integral** $\prod_a^b f(x)dx$ as follows:

If we define Δx, x_i, and x_i^* as usual, then:

$$\prod_a^b f(x)dx = \lim_{n\to\infty} \left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}$$

That is, instead of summing up the $f(x_i^*)$, we *multiply* them!

Question: Express $\prod_a^b f(x)dx$ in terms of $\int_a^b f(x)dx$

Hint: How do you turn a product into a sum?

Note: In other words, although this *looks* like a new concept, it really isn't, which is quite surprising!

Any comments about this exam? (too long? too hard?)

CONGRATULATIONS!!!

You're officially done with this course! :) Thank you so much for having me, and I hope you had a lot of fun! :)

Any other comments or goodbye words?